SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 4

Problem 1. Let X be a smooth manifold and $f : X \to \mathbb{R}$ by C^∞. Show that T^*X is a vector bundle over X. Furthermore, show that the family of functions defined by $\theta_x(v) = D_v(f)$ is a C^∞ section of T^*M, where $v \in T_xM$ and D_v is the derivation determined by v. θ_x is called the *differential of* f and is often written df.

Problem 2. Show that if $g : \mathbb{R} \to \mathbb{R}$ is a C^∞ function such that $g(z) \neq 0$ for any $z \in \mathbb{R}$, then the integral curves of the vector field $V = xg(xy)\frac{\partial}{\partial x} - yg(xy)\frac{\partial}{\partial y}$ on $\mathbb{R}^2 \setminus \{0\}$ are the level sets of the function $F(x, y) = xy$. [*Hint*: Show that the level sets of F and integral curves of V have the same tangent bundles]

Problem 3. Let $F : (-\varepsilon, \varepsilon) \times X \to X$ be any C^∞ map such that $F(t, F(s, x)) = F(t + s, x)$ whenever $|t|, |s|$ and $|t + s|$ are all less than ε.

(i) Show that $V(x) := \frac{\partial}{\partial t}|_{t=0}F(t, x)$ is a C^∞ vector field on X.
(ii) Show that F is the flow generated by V.
(iii) Show that F extends uniquely to a globally defined flow $F : \mathbb{R} \times M \to M$ satisfying the flow equation [*Hint*: If $k \in \mathbb{Z}$ and $\delta \in (0, \varepsilon)$, define $F(k\varepsilon/2 + \delta, x) = T^k(F(\delta, x))$, where $T(x) = F(\varepsilon/2, x)$].
(iv) Which assumption(s) is/are not satisfied for flows which reach the boundary in finite time?

Problem 4. Show that if X is a compact manifold and V is a smooth vector field on M, then there exists a globally defined flow φ_t^V [*Hint*: Fix a finite open cover of charts of X, and let $T(x)$ be the largest $\varepsilon > 0$ such that the flow is defined on $(-\varepsilon, \varepsilon)$ in *some* chart from the finite cover. Show that T is continuous and positive, hence bounded below. Apply the previous problem.]